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MATHEMATICAL, OR "LINEAR," PROGRAMMING: A NONMATHEMATICAL EXPOSITION

By ROBERT DORFMAN*

This paper is intended to set forth the leading ideas of mathematical programming[1] purged of the algebraic apparatus which has impeded their general acceptance and appreciation. This will be done by concentrating on the graphical representation of the method. While it is not possible, in general, to portray mathematical programming problems in two-dimensional graphs, the conclusions which we shall draw from the graphs will be of general validity and, of course, the graphic representation of multidimensional problems has a time-honored place in economics.

The central formal problem of economics is the problem of allocating scarce resources so as to maximize the attainment of some predetermined objective. The standard formulation of this problem—the so-called marginal analysis—has led to conclusions of great importance for the understanding of many questions of social and economic policy. But it is a fact of common knowledge that this mode of analysis has not recommended itself to men of affairs for the practical solution of their economic and business problems. Mathematical programming is based on a restatement of this same formal problem in a form which is designed to be useful in making practical decisions in business and economic affairs. That mathematical programming is nothing but a reformulation of the standard economic problem and its solution is the main thesis of this exposition.

The motivating idea of mathematical programming is the idea of a

* The author is associate professor of economics at the University of California, Berkeley.

[1] The terminology of the techniques which we are discussing is in an unsatisfactory state. Most frequently they are called "linear programming" although the relationships involved are not always linear. Sometimes they are called "activities analysis," but this is not a very suggestive name. The distinguishing feature of the techniques is that they are concerned with programming rather than with analysis, and, at any rate, "activities analysis" has not caught on. We now try out "mathematical programming"; perhaps it will suit.

"process" or "activity." A process is a specific method for performing an economic task. For example, the manufacture of soap by a specified formula is a process. So also is the weaving of a specific quality of cotton gray goods on a specific type of loom. The conventional production function can be thought of as the formula relating the inputs and outputs of all the processes by which a given task can be accomplished.

For some tasks, *e.g.*, soap production, there are an infinite number of processes available. For others, *e.g.*, weaving, only a finite number of processes exist. In some cases, a plant or industry may have only a single process available.

In terms of processes, choices in the productive sphere are simply decisions as to which processes are to be used and the extent to which each is to be employed. Economists are accustomed to thinking in terms of decisions as to the quantities of various productive factors to be employed. But an industry or firm cannot substitute Factor A for Factor B unless it does some of its work in a different way, that is, unless it substitutes a process which uses A in relatively high proportions for one which uses B. Inputs, therefore, cannot be changed without a change in the way of doing things, and often a fundamental change. Mathematical programming focusses on this aspect of economic choice.

The objective of mathematical programming is to determine the optimal levels of productive processes in given circumstances. This requires a restatement of productive relationships in terms of processes and a reconsideration of the effect of factor scarcities on production choices. As a prelude to this theoretical discussion, however, it will be helpful to consider a simplified production problem from a common-sense point of view.

I. *An Example of Mathematical Programming*

Let us consider an hypothetical automobile company equipped for the production of both automobiles and trucks. This company, then, can perform two economic tasks, and we assume that it has a single process for accomplishing each. These two tasks, the manufacture of automobiles and that of trucks, compete for the use of the firm's facilities. Let us assume that the company's plant is organized into four departments: (1) sheet metal stamping, (2) engine assembly, (3) automobile final assembly, and (4) truck final assembly—raw materials, labor, and all other components being available in virtually unlimited amounts at constant prices in the open market.

The capacity of each department of the plant is, of course, limited. We assume that the metal stamping department can turn out sufficient stampings for 25,000 automobiles or 35,000 trucks per month. We can then calculate the combinations of automobile and truck stampings

which this department can produce. Since the department can accommodate 25,000 automobiles per month, each automobile requires 1/25,000 or 0.004 per cent of monthly capacity. Similarly each truck requires 0.00286 per cent of monthly capacity. If, for example, 15,000 automobiles were manufactured they would require 60 per cent of metal stamping capacity and the remaining 40 per cent would be sufficient to produce stampings for 14,000 trucks. Then 15,000 automobiles and 14,000 trucks could be produced by this department at full operation. This is, of course, not the only combination of automobiles and trucks which could be produced by the stamping department at full operation. In Figure 1, the line labeled "Metal Stamping" represents all such combinations.

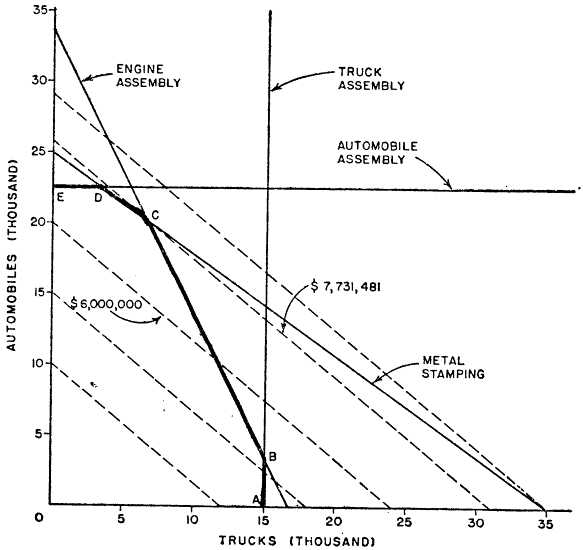

FIGURE 1. CHOICES OPEN TO AN AUTOMOBILE FIRM

Similarly we assume that the engine assembly department has monthly capacity for 33,333 automobile engines or 16,667 truck engines or, again, some combination of fewer automobile and truck engines. The combinations which would absorb the full capacity of the engine assembly department are shown by the "Engine Assembly" line in Figure 1. We assume also that the automobile assembly department can accommodate 22,500 automobiles per month and the truck assembly department 15,000 trucks. These limitations are also represented in Figure 1.

We regard this set of assumptions as defining two processes: the

production of automobiles and the production of trucks. The process of producing an automobile yields, as an output, one automobile and absorbs, as inputs, 0.004 per cent of metal stamping capacity, 0.003 per cent of engine assembly capacity, and 0.00444 per cent of automobile assembly capacity. Similarly the process of producing a truck yields, as an output, one truck and absorbs, as inputs, 0.00286 per cent of metal stamping capacity, 0.006 per cent of engine assembly capacity, and 0.00667 per cent of truck assembly capacity.

The economic choice facing this firm is the selection of the numbers of automobiles and trucks to be produced each month, subject to the restriction that no more than 100 per cent of the capacity of any department can be used. Or, in more technical phraseology, the choice consists in deciding at what level to employ each of the two available processes. Clearly, if automobiles alone are produced, at most 22,500 units per month can be made, automobile assembly being the effective limitation. If only trucks are produced, a maximum of 15,000 units per month can be made because of the limitation on truck assembly. Which of these alternatives should be adopted, or whether some combination of trucks and automobiles should be produced depends on the relative profitability of manufacturing trucks and automobiles. Let us assume, to be concrete, that the sales value of an automobile is $300 greater than the total cost of purchased materials, labor, and other direct costs attributable to its manufacture. And, similarly, that the sale value of a truck is $250 more than the direct cost of manufacturing it. Then the net revenue of the plant for any month is 300 times the number of automobiles produced plus 250 times the number of trucks. For example, 15,000 automobiles and 6,000 trucks would yield a net revenue of $6,000,000. There are many combinations of automobiles and trucks which would yield this same net revenue; 10,000 automobiles and 12,000 trucks is another one. In terms of Figure 1, all combinations with a net revenue of $6,000,000 lie on a straight line, to be specific, the line labelled $6,000,000 in the figure.

A line analogous to the one which we have just described corresponds to each possible net revenue. All these lines are parallel, since their slope depends only on the relative profitability of the two activities. The greater the net revenue, of course, the higher the line. A few of the net revenue lines are shown in the figure by the dashed parallel lines.

Each conceivable number of automobiles and trucks produced corresponds to a point on the diagram, and through each point there passes one member of the family of net revenue lines. Net revenue is maximized when the point corresponding to the number of automobiles and trucks produced lies on the highest possible net revenue line. Now the effect of the capacity restrictions is to limit the range of choice to

outputs which correspond to points lying inside the area bounded by the axes and by the broken line ABCDE. Since net revenue increases as points move out from the origin, only points which lie on the broken line need be considered. Beginning then with Point A and moving along the broken line we see that the boundary of the accessible region intersects higher and higher net revenue lines until point C is reached. From there on, the boundary slides down the scale of net revenue lines. Point C therefore corresponds to the highest attainable net revenue. At point C the output is 20,370 automobiles and 6,481 trucks, yielding a net revenue of $7,731,481 per month.

The reader has very likely noticed that this diagram is by no means novel. The broken line, ABCDE, tells the maximum number of automobiles which can be produced in conjunction with any given number of trucks. It is therefore, apart from its angularity, a production opportunity curve or transformation curve of the sort made familiar by Irving Fisher, and the slope of the curve at any point where it has a slope is the ratio of substitution in production between automobiles and trucks. The novel feature is that the production opportunity curve shown here has no defined slope at five points and that one of these five is the critical point. The dashed lines in the diagram are equivalent to conventional price lines.

The standard theory of production teaches that profits are maximized at a point where a price line is tangent to the production opportunity curve. But, as we have just noted, there are five points where our production opportunity curve has no tangent. The tangency criterion therefore fails. Instead we find that profits are maximized at a corner where the slope of the price line is neither less than the slope of the opportunity curve to the left of the corner nor greater than the slope of the opportunity curve to the right.

Diagrammatically, then, mathematical programming uses angles where standard economics uses curves. In economic terms, where does the novelty lie? In standard economic analysis we visualize production relationships in which, if there are two products, one may be substituted for the other with gradually increasing difficulty. In mathematical programming we visualize a regime of production in which, for any output, certain factors will be effectively limiting but other factors will be in ample supply. Thus, in Figure 1, the factors which effectively limit production at each point can be identified by noticing on which limitation lines the point lies. The rate of substitution between products is determined by the limiting factors alone and changes only when the designation of the limiting factors changes. In the diagram a change in the designation of the limiting factors is represented by turning a corner on the production opportunity curve.

We shall come back to this example later, for we have not exhausted its significance. But now we are in a position to develop with more generality some of the concepts used in mathematical programming.

II. *The Model of Production in Mathematical Programming*

A classical problem in economics is the optimal utilization of two factors of production, conveniently called capital and labor. In the usual analysis, the problem is formulated by conceiving of the two factors as cooperating with each other in accordance with a production function which states the maximum quantity of a product which can be obtained by the use of stated quantities of the two factors. One convenient means of representing such a production function is an "isoquant diagram," as in Figure 2. In this familiar figure, quantities of labor are plotted along the horizontal axis and quantities of capital

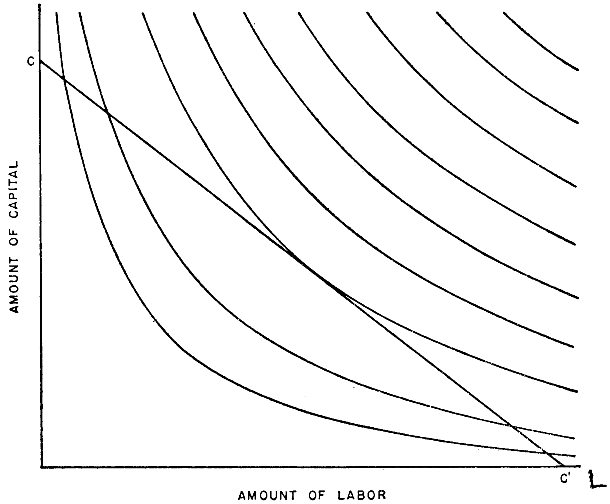

FIGURE 2. AN ISOQUANT DIAGRAM

along the vertical. Each of the arcs in the body of the diagram corresponds to a definite quantity of output, higher arcs corresponding to greater quantities.

If the prices per unit of capital and labor are known, the combinations of labor and capital which can be purchased for a fixed total expenditure can be shown by a sloping straight line like CC′ in the figure, the slope depending only on the relative prices. Two interpretations follow immediately. First, the minimum unit cost of producing the output represented by any isoquant can be achieved by using the combination of labor and capital which corresponds to the point where that

isoquant is tangent to a price line. Second, the greatest output attainable with any given expenditure is represented by the isoquant which is tangent to the price line corresponding to that expenditure.

This diagram and its analysis rest upon the assumption that the two factors are continuously substitutable for each other in such wise that if the amount of labor employed be reduced by a small amount it will be possible to maintain the quantity of output by a *small* increase in the amount of capital employed. Moreover, this analysis assumes that each successive unit decrement in the amount of labor will require a slightly larger increment in the amount of capital if output is to remain constant. Otherwise the isoquants will not have the necessary shape.

All this is familiar. We call it to mind only because we are about to develop an analogous diagram which is fundamental to mathematical programming. First, however, let us see why a new diagram and a new approach are felt to be necessary.

The model of production which we have just briefly sketched very likely is valid for some kinds of production. But for most manufacturing industries, and indeed all production where elaborate machinery is used, it is open to serious objection. It is characteristic of most modern machinery that each kind of machine operates efficiently only over a narrow range of speeds and that the quantities of labor, power, materials and other factors which cooperate with the machine are dictated rather inflexibly by the machine's built-in characteristics. Furthermore, at any time there is available only a small number of different kinds of machinery for accomplishing a given task. A few examples may make these considerations more concrete. Earth may be moved by hand shovels, by steam or diesel shovels, or by bulldozers. Power shovels and bulldozers are built in only a small variety of models, each with inherent characteristics as to fuel consumption per hour, number of operators and assistants required, cubic feet of earth moved per hour, etc. Printing type may be set by using hand-fonts, linotype machines or monotype machines. Again, each machine is available in only a few models and each has its own pace of operation, power and space requirements, and other essentially unalterable characteristics. A moment's reflection will bring to mind dozens of other illustrations: printing presses, power looms, railroad and highway haulage, statistical and accounting calculation, metallic ore reduction, metal fabrication, etc. For many economic tasks the number of processes available is finite, and each process can be regarded as inflexible with regard to the ratios among factor inputs and process outputs. Factors cannot be substituted for each other except by changing the levels at which entire technical processes are used, because each process uses factors in fixed characteristic ratios. In mathematical programming, accordingly, process substi-

tution plays a rôle analogous to that of factor substitution in conventional analysis.

We now develop an apparatus for the analysis of process substitution. For convenience we shall limit our discussion to processes which consume two factors, to be called capital and labor, and produce a single output. Figure 3 represents such a process. As in Figure 2, the horizontal axis is scaled in units of labor and the vertical axis in units of capital. The process is represented by the ray, OA, which is scaled in units of output. To each output there corresponds a labor requirement found by locating the appropriate mark on the process ray and reading straight down. The capital requirement is found in the same manner by reading straight across from the mark on the process line. Similarly, to each amount of labor there corresponds a quantity of out-

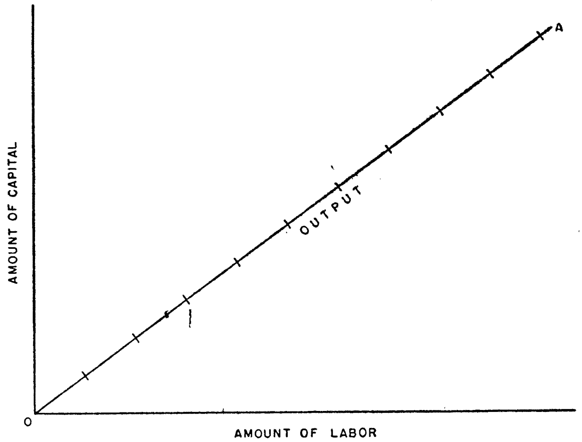

FIGURE 3. A PROCESS

put, found by reading straight up, and a quantity of capital, found by reading straight across from the output mark.

It should be noted that the quantity of capital in this diagram is the quantity used in a process rather than the quantity owned by an economic unit; it is capital-service rather than capital itself. Thus, though more or less labor may be combined with a given machine—by using it more or fewer hours—the ratio of capital to labor inputs, that is, the ratio of machine-hours to labor hours—is regarded as technologically fixed.

Figure 3 incorporates two important assumptions. The fact that the line OA is straight implies that the ratio between the capital input and the labor input is the same for all levels of output and is given, indeed, by the slope of the line. The fact that the marks on the output line are

evenly spaced indicates that there are neither economies nor diseconomies of scale in the use of the process, *i.e.*, that there will be strict proportionality between the quantity of output and the quality of either input. These assumptions are justified rather simply on the basis of the notion of a process. If a process can be used once, it can be used twice or as many times as the supplies of factors permit. Two linotype machines with equally skilled operators can turn out just twice as much type per hour as one. Two identical mills can turn out just twice as many yards of cotton per month as one. So long as factors are available, a process can be duplicated. Whether it will be economical to do so is, of course, another matter.

If there is only one process available for a given task there is not much scope for economic choice. Frequently, however, there will be

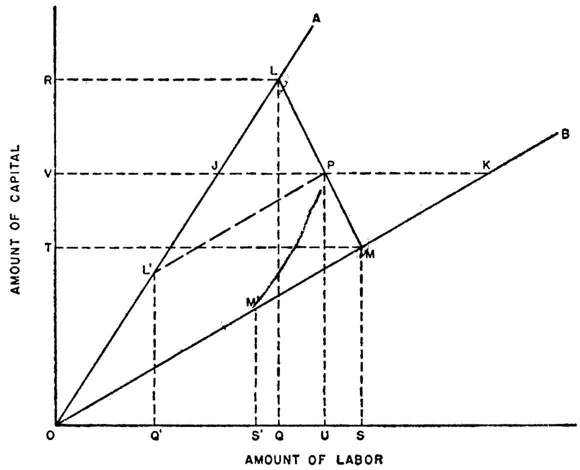

FIGURE 4. TWO PROCESSES

several processes. Figure 4 represents a situation in which two procedures are available, Process A indicated by the line OA and Process B indicated by OB. We have already seen how to interpret points on the lines OA and OB. The scales by which output is measured on the two rays are not necessarily the same. The scale on each ray reflects the productivity of the factors when used in the process represented by that ray and has no connection with the output scale on any other process ray. Now suppose that points L and M represent production of the same output by the two processes. Then LM, the straight line between them, will represent an isoquant and each point on this line will correspond to a combination of Processes A and B which produces the same output as OL units of Process A or OM units of Process B.

To see this, consider any point P on the line LM and draw a line

through P parallel to OB. Let L′ be the point where this line intersects OA. Finally mark the point M′ on OB such that OM′ = L′P. Now consider the production plan which consists of using Process A at level OL′ and Process B at level OM′.[2] It is easy to show that this production plan uses OU units of labor, where U is the labor coordinate of point P, and OV units of capital, where V is the capital coordinate of point P.[3]

Since the coordinates of point P correspond to the quantities of factors consumed by OL′ units of Process A and OM′ units of Process B, we interpret P as representing the combined production plan made up of the specified levels of the two processes. This interpretation implies an important economic assumption, namely, that if the two processes are used simultaneously they will neither interfere with nor enhance each other so that the inputs and outputs resulting from simultaneous use of two processes at any levels can be found by adding the inputs and outputs of the individual processes.

In order to show that P lies on the isoquant through points L and M it remains only to show that the sum of the outputs corresponding to points L′ and M′ is the same as the output corresponding to point L or point M. This follows at once from the facts that the output corresponding to any point on a process ray is directly proportional to the length of the ray up to that point and that the triangles LL′P and LOM in Figure 4 are similar.[4] Thus if we have two process lines like OA and OB and find points L and M on them which represent producing the same output by means of the two processes then the line segment connecting the two equal-output points will be an isoquant.

We can now draw the mathematical programming analog of the familiar isoquant diagram. Figure 5 is such a diagram with four process lines shown. Point M represents a particular output by use of Process A and points L, K, J represent that same output by means of Processes B, C, D, respectively. The succession of line segments connecting these

[2] An alternative construction would be to draw a line through point P parallel to OA. It would intersect OB at M′. Then we could lay off OL′ equal to M′P on OA. This would lead to exactly the same results as the construction used in the text. The situation is analogous to the "parallelogram of forces" in physics.

[3] Proof: Process A at level OL′ uses OQ′ units of labor, Process B at level OM′ uses OS′ units of labor, together they use OQ′ + OS′ units of labor. But, by construction, L′P is equal and parallel to OM′. So Q′U = OS′. Therefore, OQ′ + OS′ = OQ′ + Q′U = OU units of labor. The argument with respect to capital is similar.

[4] Proof: Let Output (X) denote the output corresponding to any point, X, on the diagram. Then Output (M′)/Output (M) = OM′/OM and Output (L′)/Output (L) = OL′/OL. By assumption: Output (L) = Output (M). So Output (M′)/Output (L) = OM′/OM. Adding, we have:

$$\frac{\text{Output }(M') + \text{Output }(L')}{\text{Output }(L)} = \frac{OM'}{OM} + \frac{OL'}{OL} = \frac{L'P}{OM} + \frac{OL'}{OL} = \frac{L'L}{OL} + \frac{OL'}{OL} = 1.$$

four points is the isoquant for that same output. It is easy to see that any other succession of line segments respectively parallel to those of MLKJ is also an isoquant. Three such are shown in the figure. It is instructive to compare Figure 5 with Figure 2 and note the strong resemblance in appearance as well as in interpretation.

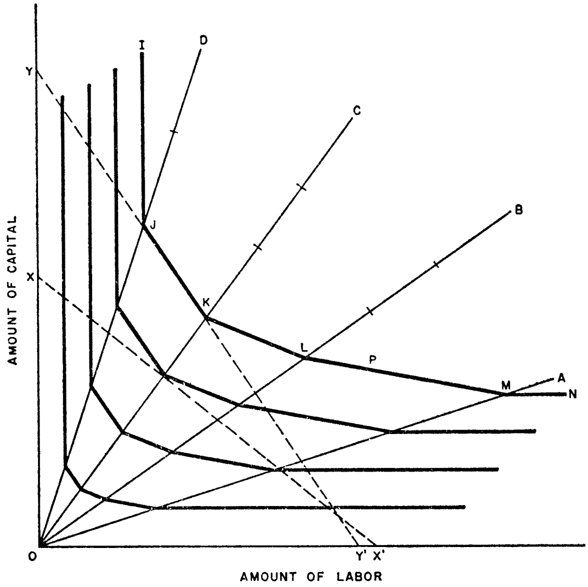

FIGURE 5. FOUR PROCESSES

We may draw price lines on Figure 5, just as on the conventional kind of isoquant diagram. The dashed lines XX′ and YY′ represent two possible price lines. Consider XX′ first. As that line is drawn, the maximum output for a given expenditure can be obtained by use of Process C alone, and, conversely, the minimum cost for a given output is also obtained by using Process C alone. Thus, for the relative price regime represented by XX′, Process C is optimal. The price line YY′ is drawn parallel to the isoquant segment JK. In this case Process C is still optimal, but Process D is also optimal and so is any combination of the two.

It is evident from considering these two price lines, and as many others as the reader wishes to visualize, that an optimal production program can always be achieved by means of a single process, which process depending, of course, on the slope of the price line. It should be noted, however, that the conventional tangency criterion is no longer applicable.

We found in Figure 5 that an optimal economic plan need never use

more than a single process for each of its outputs.[5] That conclusion is valid for the situation depicted, which assumed that the services of the two factors could be procured in any amounts desired at constant relative prices. This assumption is not applicable to many economic problems, nor is it used much in mathematical programming. We must now, therefore, take factor supply conditions into account.

III. *Factor Supplies and Costs*

In mathematical programming it is usual to divide all factors of production into two classes: unlimited factors, which are available in any amount desired at constant unit cost, and limited or scarce factors, which are obtainable at constant unit cost up to a fixed maximum quantity and thereafter not at all. The automobile example illustrates this classification. There the four types of capacity were treated as fixed factors available at zero variable cost; all other factors were grouped under direct costs which were considered as constant per unit of output.

The automobile example showed that this classification of factors is adequate for expressing the maximization problem of a firm dealing in competitive markets. In the last section we saw that when all factors are unlimited, this formulation can be used to find a minimum average cost point.

Both of these applications invoked restrictive assumptions and, furthermore, assumptions which conflict with those conventionally made in studying resource allocation. In conventional analysis we conceive that as the level of production of a firm, industry or economy rises, average unit costs rise also after some point. The increase in average costs is attributable in part to the working of the law of variable proportions,[6] which operates when the inputs of some but not all factors of production are increased. As far as the consequences of increasing some but not all inputs are concerned, the contrast between mathematical programming and the marginal analysis is more verbal than substantive. A reference to Figure 4 will show how such changes are handled in mathematical programming. Point J in Figure 4 represents the production of a certain output by the use of Process A alone. If it is desired to increase output without increasing the use of capital, this can be done by moving to the right along the dotted line JK, since this line cuts successively higher isoquants. Such a movement would correspond to using increasingly more of Process B and increasingly

[5] Recall, however, that we have not taken joint production into account nor have we considered the effects of considerations from the demand side.

[6] *Cf.* J. M. Cassels, "On the Law of Variable Proportions," in W. Fellner and B. F. Haley, eds., *Readings in the Theory of Income Distribution* (Philadelphia, 1946), pp. 103-18.

less of Process A and thus, indirectly, to substituting labor for capital. If, further, we assume that unit cost of production is lower for Process A than for Process B this movement would also correspond to increasing average cost of production. Thus both marginal analysis and mathematical programming lead to the same conclusion when factor proportions are changed: if the change starts from a minimum cost point the substitution will lead to gradually increasing unit costs.

But changing input proportions is only one part of the story according to the conventional type of analysis. If output is to be increased, any of three things may happen. First, it may be possible to increase the consumption of all inputs without incurring a change in their unit prices. In this case both mathematical programming and marginal analysis agree that output will be expanded without changing the ratios among the input quantities and average cost of production will not increase.[7] Second, it may not be possible to increase the use of some of the inputs. This is the case we have just analyzed. According to both modes of analysis the input ratios will change in this case and average unit costs will increase. The only difference between the two approaches is that if average cost is to be plotted against output, the marginal analyst will show a picture with a smoothly rising curve while the mathematical programmer will show a broken line made up of increasingly steep line segments. Third, it may be possible to increase the quantities of all inputs but only at the penalty of increasing unit prices or some kind of diseconomies of scale. This third case occurs in the marginal analysis, indeed it is the case which gives long-run cost curves their familiar shape, but mathematical programming has no counterpart for it.

The essential substantive difference we have arrived at is that the marginal analysis conceives of pecuniary and technical diseconomies associated with changes in scale while mathematical programming does not.[8] There are many important economic problems in which factor prices and productivities do not change in response to changes in scale or in which such variations can be disregarded. Most investigations of industrial capacity, for example, are of this nature. In such studies we seek the maximum output of an industry, regarding its inventory of physical equipment as given and assuming that the auxiliary factors needed to cooperate with the equipment can be obtained in the quanti-

[7] *Cf.* F. H. Knight, *Risk, Uncertainty and Profit* (Boston, 1921), p. 98.

[8] Even within the framework of the marginal analysis the concept of diseconomies of scale has been challenged on both theoretical and empirical grounds. For examples of empirical criticism see Committee on Price Determination, Conference on Price Research, *Cost Behavior and Price Policy* (New York, 1943). The most searching theoretical criticism is in Piero Sraffa, "The Laws of Returns under Competitive Conditions," *Econ. Jour.*, Dec. 1926, XXXVI, 535-50.

ties dictated by the characteristics of the equipment. Manpower requirement studies are of the same nature. In such studies we take both output and equipment as given and calculate the manpower needed to operate the equipment at the level which will yield the desired output. Studies of full employment output fall into the same format. In such studies we determine in advance the quantity of each factor which is to be regarded as full employment of that factor. Then we calculate the optimum output obtainable by the use of the factors in those quantities.

These illustrations should suffice to show that the assumptions made in mathematical programming can comprehend a wide variety of important economic problems. The most useful applications of mathematical programming are probably to problems of the types just described, that is, to problems concerned with finding optimum production plans using specified quantities of some or all of the resources involved.

IV. *Analysis of Production with Limited Factors*

The diagrams which we have developed are readily adaptable to the analysis of the consequences of limits on the factor supplies. Such limits are, of course, the heart of Figure 1 where the four principal lines represent limitations on the process levels which result from limits on the four factor quantities considered. But Figure 1 cannot be used when more than two processes have to be considered. For such problems diagrams like Figures 3, 4, and 5 have to be used.

Figure 6 reproduces the situation portrayed in Figure 5 with some

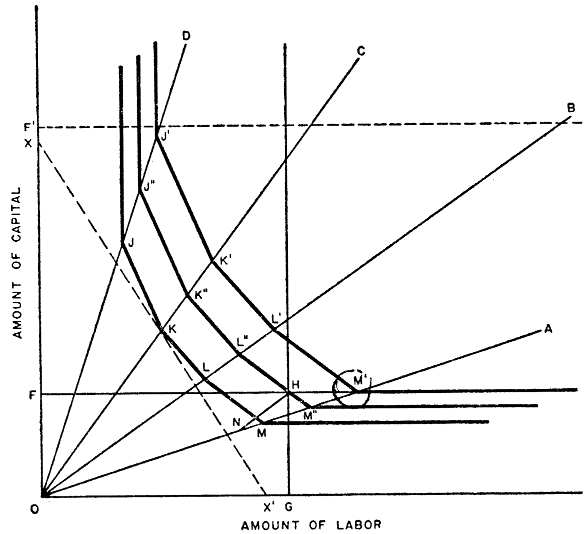

FIGURE 6. FOUR PROCESSES, WITH LIMITATIONS

additional data to be explained below. Let OF represent the maximum amount of capital which can be used and thus show a factor limitation. The horizontal line through F divides the diagram into two sections: all points above the line correspond to programs which require more capital than is available; points on and below the line represent programs which do not have excessive capital requirements. This horizontal line will be called the capital limitation line. Points on or below it are called "feasible," points above it are called "infeasible."

The economic unit portrayed in Figure 6 has the choice of operating at any feasible point. If maximum output is its objective, it will choose a point which lies on the highest possible isoquant, *i.e.*, the highest isoquant which touches the capital limitation line. This is the one labelled J'K'L'M', and the highest possible output is attained by using Process A.

Of course, maximum output may not be the objective. The objective may be, for example, to maximize the excess of the value of output over labor costs. We shall refer to such an excess as a "net value." The same kind of diagram can be used to solve for a net value provided that the value of each unit of output is independent of the number of units produced[9] and that the cost of each unit of labor is similarly constant. If these provisos are met, each point on a process ray will correspond to a certain physical output but also to a certain value of output, cost of labor, and net value of output. Further, along any process ray the net value of output will equal the physical output times the net value per unit and will therefore be proportional to the physical output. We may thus use a diagram similar to Figure 6 except that we think of net value instead of physical output as measured along the process rays and we show isovalue line instead of isoquants. This has been done on Figure 7, in which the maximum net value attainable is the one which corresponds to the isovalue contour through point P, and is attained by using Process C.

It should be noted in both Figures 6 and 7 that the optimal program consisted of a single process, that shifts in the quantity of capital available would not affect the designation of the optimal process though they would change its level, and that the price lines, which were crucial in Figure 5, played no rôle.

The next complication, and the last one we shall be able to consider, is to assume that both factors are in limited supply. This situation is portrayed in Figure 6 by adding the vertical line through point G to represent a labor limitation. The available quantity of labor is shown, of course, by the length OG. Then the points inside the rectangle

[9] This is a particularly uncomfortable assumption. We use it here to explain the method in its least complicated form.

OFHG represent programs which can be implemented in the sense that they do not require more than the available supplies of either factor. This is the rectangle of feasible programs. The greatest achievable output is the one which corresponds to the highest isoquant which touches the rectangle of feasible programs. This is the isoquant J″K″L″M″, and furthermore, since the maximum isoquant touches the rectangle at H, H represents the program by which the maximum output can be produced.

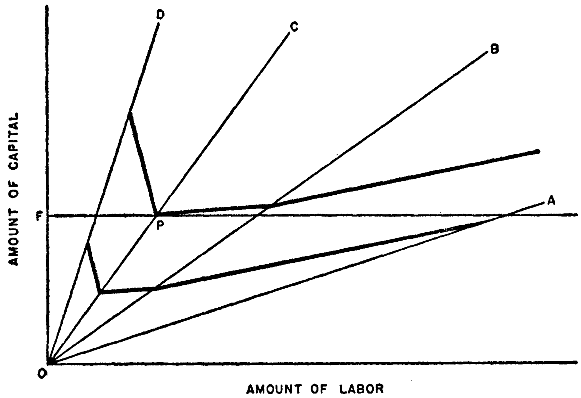

FIGURE 7. FOUR PROCESSES WITH ISOVALUE LINES

This solution differs from the previous ones in that the solution-point does not lie on any process ray but between the rays for Processes A and B. We have already seen that a point like H represents using Process A at level ON and Process B at level NH.

Two remarks are relevant to this solution. First: with the factor limitation lines as drawn, the maximum output requires two processes. If the factor limitation lines had been drawn so that they intersected exactly on one of the process rays, only one process would have been required. If the factor limitation lines had crossed to the left of Process D or to the right of Process A, the maximizing production plan would require only one process. But, no matter how the limitation lines be drawn, at most two processes are required to maximize output. We are led to an important generalization: maximum output may always be obtained by using a number of processes which does not exceed the number of factors in limited supply, if this number is greater than zero. The conclusions we drew from Figures 6 and 7 both conform to this rule, and it is one of the basic theorems of mathematical programming.

Second: although at most two processes are required to obtain the maximum output, which two depends on the location of the factor limits. As shown, the processes used for maximum output were Proces-

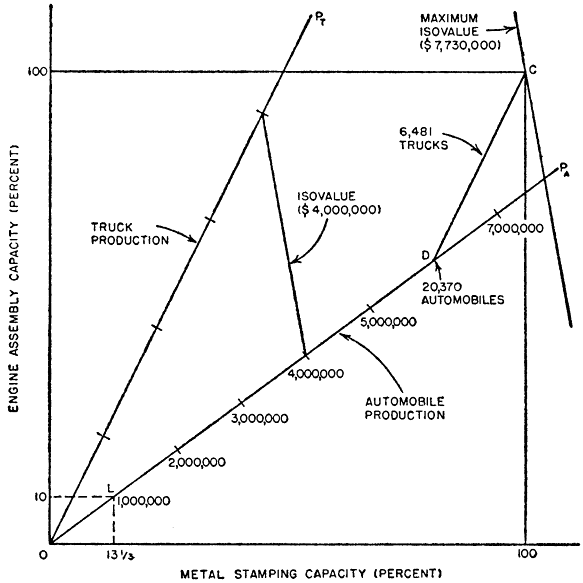

FIGURE 8. AUTOMOBILE EXAMPLE, OPTIMAL PLAN

ses A and B. If somewhat more capital, represented by the amount OF′, were available, the maximizing processes would have been Processes C and D. If two factors are limited, it is the ratio between their supplies rather than the absolute supplies of either which determines the processes in the optimum program. This contrasts with the case in which only one factor is limited. Just as the considerations which determine the optimum set of processes are more complicated when two factors are limited than when only one is, so with three or more limited factors the optimum conditions become more complicated still and soon pass the reach of intuition. This, indeed, is the *raison d'être* of the formidable apparatus of mathematical programming.

We can make these considerations more concrete by applying them to the automobile example. Referring to Figure 1, (p. 799), we note that the optimum production point, C, lay on the limitation lines for engine assembly and metal stamping, but well below the limits for automobile and truck assembly. The limitations on automobile and truck assembly capacity are, therefore, ineffective and can be disregarded. The situation in terms of the two effectively limiting types of capacity is shown in Figure 8.

In Figure 8 the ray P_A represents the process of producing automobiles and P_T the process of producing trucks. These two processes can be operated at any combination of levels which does not require the use of more than 100 per cent of either metal stamping or engine assembly

capacity. Thus the rectangle in the diagram is the region of feasible production programs. The optimal production program is the one in the feasible region which corresponds to the highest possible net revenue.[10] Thus it will be helpful to construct isorevenue lines, as we did in Figure 7. To do this, consider automobile production first. Each point on P_A corresponds to the production of a certain number of automobiles per month. Suppose, for example, that the scale is such that point L represents the production of 3,333 automobiles per month. It will be recalled that each automobile yields a net revenue of $300. Therefore, 3,333 automobiles yield a revenue of $1,000,000. Point L, then, corresponds to a net revenue of $1,000,000 as well as to an output of 3,333 automobiles per month. Since (see page 799), 3,333 automobiles require 13⅓ per cent of metal stamping capacity and 10 per cent of engine assembly capacity, the coordinates of the $1,000,000 net revenue point on P_A are established at once. By a similar argument, the point whose coordinates are 26⅔ per cent of metal stamping capacity and 20 per cent of engine capacity is the $2,000,000 net revenue point on P_A. In the same manner, the whole ray can be drawn and scaled off in terms of net revenue, and so can P_T, the process ray for truck production. The diagram is completed by connecting the $4,000,000 points on the two process lines in order to show the direction of the isorevenue lines.

The optimum program is at point C, where the two capacity limits intersect, because C lies on the highest isorevenue line which touches the feasible region. Through point C we have drawn a line parallel to the truck production line and meeting the automobile production line at D. By our previous argument, the length OD represents the net revenue from automobile production in the optimal program and the length DC represents the net revenue from trucks. If these lengths be scaled off, the result, of course, will be the same as the solution found previously.

V. *Imputation of Factor Values*

We have just noted that the major field of application of mathematical programming is to problems where the supply of one or more factors of production is absolutely limited. Such scarcities are the genesis of value in ordinary analysis, and they generate values in mathematical programming too. In fact, in ordinary analysis the determination of outputs and the determination of prices are but two aspects of the same problem, the optimal allocation of scarce resources. The same is true in mathematical programming.

[10] Since the objective of the firm is, by assumption, to maximize revenue rather than physical output, we may consider automobile and truck production as two alternative processes for producing revenue instead of as two processes with disparate outputs.

Heretofore we have encountered prices only as data for determining the direct costs of processes and the net value of output. But of course the limiting factors of production also have value although we have not assigned prices to them up to now. In this section we shall see that the solution of a mathematical programming problem implicitly assigns values to the limiting factors of production. Furthermore, the implicit pricing problem can be solved directly and, when so solved, constitutes a solution to the optimal allocation problem.

Consider the automobile example and ask: how much is a unit (1 per cent) of each of the types of capacity worth to the firm? The approach to this question is similar in spirit to the familiar marginal analysis. With respect to each type of capacity we calculate how much the maximum revenue would increase if one more unit were added, or how much revenue would decrease if one unit were taken away. Since there is a surplus of automobile assembly capacity, neither the addition nor the subtraction of one unit of this type would affect the optimum program or the maximum net revenue. Hence the value of this type of capacity is nil. The analysis and result for truck assembly are the same.

We find, then, that these two types of capacity are free goods. This does not imply that an automobile assembly line is not worth having, any more than, to take a classic example, the fact that air is a free good means that it can be dispensed with. It means that it would not be worth while to increase this type of capacity at any positive price and that some units of these types could be disposed of without loss.

The valuation of the other types of capacity is not so trivial. In Figure 9 possible values per per cent of engine assembly capacity are scaled along the horizontal axis and values per per cent of metal stamping capacity are scaled along the vertical axis. Now consider any possible pair of values, say engine assembly capacity worth \$20,000 per unit and metal stamping worth \$40,000. This is represented by point A on the figure. Applying these values to the data on pages 798-99, the values of capacity required for producing an automobile is found to be: $(0.004 \times \$40{,}000) + (0.003 \times \$20{,}000) = \$220$ which is well under the value of producing an automobile, or \$300.[11] In the same way, if engine assembly capacity is worth \$60,000 per per cent of capacity and metal stamping capacity is valued at \$30,000 per unit (point B), the cost of scarce resources required to produce an automobile will be exactly equal to the value of the product. This is clearly not the only combination of resource values which will precisely absorb the value of output when the resources are used to produce automobiles. The automobile production line in the figure, which passes through point B, is

[11] These unit values are also marginal values since costs of production are constant.

the locus of all such value combinations. A similar line has been drawn for truck production to represent those combinations of resource values for which the total value of resources used in producing trucks is equal to the value of output. The intersection of these two lines is obviously the only pair of resource values for which the marginal resource cost of producing an additional automobile is equal to the net value of an automobile and the same is true with respect to trucks. The pair can be found by plotting or, with more precision, by algebra. It is found that 1 per cent of engine assembly capacity is worth \$9,259 and 1 per cent of metal stamping capacity is worth \$68,056.

To each pair of values for the two types of capacity, there corresponds a value for the entire plant. Thus to the pair of values represented by point A there corresponds the plant value of $(100 \times \$20{,}000) + (100 \times \$40{,}000) = \$6{,}000{,}000$. This is not the only pair of resource values which give an aggregate plant value of \$6,000,000. Indeed, any pair of resource values on the dotted line through A corresponds to the same aggregate plant value. (By this stage, Figure 9 should become strongly reminiscent of Figure 1, page 799.) We have drawn a number of dotted lines parallel to the one just described, each corresponding to a specific aggregate plant value. The dotted line which passes through the intersection of the two production lines is of particular interest. By measurement or otherwise this line can be found to correspond to a plant value of \$7,731,500 which, we recall, was found to be the maximum attainable net revenue.

Let us consider the implications of assigning values to the two limiting factors from a slightly different angle. We have seen that as soon as unit values have been assigned to the factors an aggregate value is assigned to the plant. We can make the aggregate plant value as low as we please, simply by assigning sufficiently low values to the various factors. But if the values assigned are too low, we have the unsatisfactory consequence that some of the processes will give rise to unimputed surpluses. We may, therefore, seek the lowest aggregate plant value which can be assigned and still have no process yield an unimputed surplus. In the automobile case, that value is \$7,731,500. In the course of finding the lowest acceptable plant value we find specific unit values to be assigned to each of the resources.

In this example there are two processes and four limited resources. It turns out that only two of the resources were effectivly limiting, the others being in relatively ample supply. In general, the characteristics of the solution to a programming problem depend on the relationship between the number of limited resources and the number of processes taken into consideration. If, as in the present instance, the number of limited resources exceeds the number of processes it will usually turn

out that some of the resources will have imputed values of zero and that the number of resources with positive imputed values will be equal to the number of processes.[12] If the number of limited resources equals the number of processes all resources will have positive imputed values. If, finally, the number of processes exceeds the number of limited resources, some of the processes will not be used in the optimal program. This situation, which is the usual one, was illustrated in Figure 6. In this case the total imputed value of resources absorbed will equal net revenue for some processes and will exceed it for others. The number of

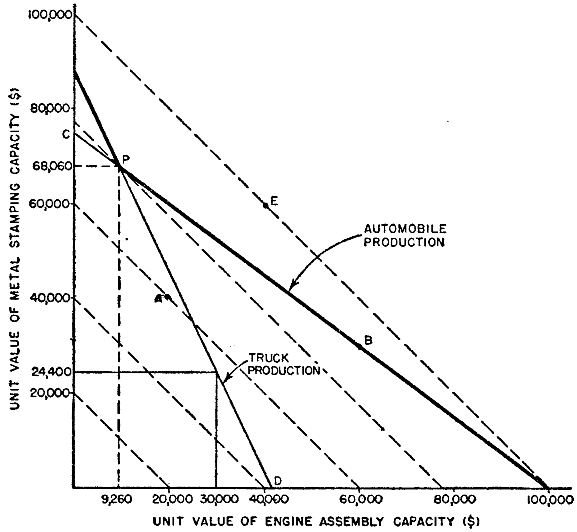

FIGURE 9. AUTOMOBILE EXAMPLE, IMPLICIT VALUES

processes for which the imputed value of resources absorbed equals the net revenue will be just equal to the number of limited resources and the processes for which the equality holds are the ones which will appear at positive levels in the optimal program. In brief, the determination of the minimum acceptable plant value amounts to the same thing as the determination of the optimal production program. The programming problem and the valuation problem are not only closely related, they are basically the same.

This can be seen graphically by comparing Figures 1 and 9. Each figure contains two axes and two diagonal boundary lines. But the boundary lines in Figure 9 refer to the same processes as the axes in Figure 1, and the axes in Figure 9 refer to the same resources as the

[12] We say "usually" in this sentence because in some special circumstances the number of resources with positive imputed values may exceed the number of processes.

diagonal boundary lines in Figure 1. Furthermore, in using Figure 1 we sought the net revenue corresponding to the highest dashed line touched by the boundary; in using Figure 9 we sought the aggregate value corresponding to the lowest dashed line which has any points on or outside the boundary; and the two results turned out to be the same. Formally stated, these two figures and the problems they represent are *duals* of each other.

The dualism feature is a very useful property in the solution of mathematical programming problems. The simplest way to see this is to note that when confronting a mathematical programming problem we have the choice of solving the problem or its dual, whichever is easier. Either way we can get the same results. We can use this feature now to generalize our discussion somewhat. Up to now when dealing with more than two processes we have had to use relatively complicated

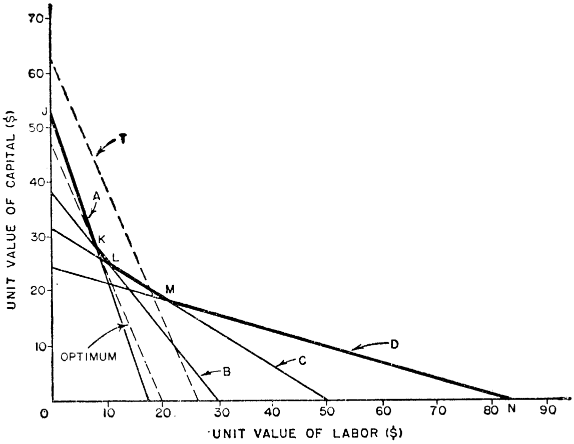

FIGURE 10. THE VALUATION PROBLEM, FOUR PROCESSES

diagrams like Figure 6 because straightforward diagrams like Figure 1 did not contain enough axes to represent the levels of the processes. Now we can use diagrams modeled on Figure 9 to depict problems with any number of processes so long as they do not involve more than two scarce factors. Figure 10 illustrates a diagram for four processes and is, indeed, derived from Figure 6. In Figure 10 line A represents all pairs of factor values such that Process A would yield neither a profit nor a loss. Lines B, C, and D are similarly interpreted. The dashed line T is a locus along which the aggregate value of the labor and capital available to the firm (or industry) is constant. Its position is not relevant to the analysis; its slope, which is simply the ratio of the quantity of available labor to that of capital, is all that is significant. The broken

line JKLMN divides the graph into two regions. All points on or above it represent pairs of resource values such that no process gives rise to an unimputed surplus. Let us call this the acceptable region. For each point below that broken line there is at least one process which does have an unimputed surplus. This is the unacceptable region. We then seek for that point in the acceptable region which corresponds to the lowest aggregate plant value. This point will, of course, give the set of resource values which makes the accounting profit of the firm as great as possible without giving rise to any unimputed income. The point which meets these requirements is K, and a dotted line parallel to T has been drawn through it to indicate the minimum acceptable aggregate plant value.

At point K Processes A and B yield zero profits, and Processes C and D yield losses. Hence Processes A and B are the ones which should be used, exactly as we found in Figure 6. To be sure, this diagram does not tell the levels at which A and B should be used, any more than Figure 6 tells the valuations to be placed on the two resources. But finding the levels after the processes have been selected is a comparatively trivial matter. All that is necessary is to find the levels which will fully utilize the resources which are not free goods. This may be done algebraically or by means of a diagram like Figure 8.

VI. *Applications*

In the first section we asserted that the principal motivation of mathematical programming was the need for a method of analysis which lent itself to the practical solution of the day-to-day problems of business and the economy in general. Immediately after making that claim we introduced a highly artificial problem followed by a rather extended discussion of abstract and formal relationships. The time has now come to indicate the basis for saying that mathematical programming is a practical method of analysis.

The essential simplification achieved in mathematical programming is the replacement of the notion of the production function by the notion of the process. The process is a highly observable unit of activity and the empirical constants which characterize it can be estimated without elaborate analysis. Furthermore in many industries the structure of production corresponds to operating a succession of processes, as we have conceived them. Many industrial decisions, like shutting down a bank of machines or operating an extra shift, correspond naturally to our concept of choosing the level of operation of a process. In brief, mathematical programming is modelled after the actual structure of production in the hope that thereby it will involve only observable

constants and directly controllable variables.

Has this hope been justified? The literature already contains a report of a successful application to petroleum refining.[13] I have made a similar application which, perhaps, will bear description. The application was to a moderate-sized refinery which produces premium and regular grades of automotive gasoline. The essential operation studied was blending. In blending, ten chemically distinct kinds of semirefined oil, called blending stocks, are mixed together. The result is a saleable gasoline whose characteristics are approximately the weighted average of the characteristics of the blending stocks. For example, if 500 gallons of a stock with octane rating of 80 are blended with 1,000 gallons of a stock with octane rating of 86 the result will be $500 + 1{,}000 = 1{,}500$ gallons of product with octane rating of $(\frac{1}{3} \times 80) + (\frac{2}{3} \times 86) = 84$.

The significant aspect of gasoline blending for our present purposes is that the major characteristics of the blend—its knock rating, its vapor pressure, its sulphur content, etc.—can be expressed as linear functions of the quantities of the various blending stocks used. So also can the cost of the blend if each of the blending stocks has a definite price per gallon. Thus the problem of finding the minimum cost blend which will meet given quality specifications is a problem in mathematical programming.

Furthermore, in this refinery the quantities of some of the blending stocks are definitely limited by contracts and by refining capacity. The problem then arises: what are the most profitable quantities of output of regular and premium gasoline, and how much of each blending stock should be used for each final product. This problem is analogous to the artificial automobile example, with the added complication of the quality specifications. The problem is too complicated for graphic analysis but was solved easily by arithmetical procedures. As far as is known, mathematical programming provides the only way for solving such problems. Charnes and Cooper have recently published the solution to a similar problem which arose in the operations of a metal-working firm.[14]

An entirely different kind of problem, also amenable to mathematical programming, arises in newsprint production. Freight is a major element in the cost of newsprint. One large newsprint company has six mills, widely scattered in Canada, and some two hundred customers, widely scattered in the United States. Its problem is to decide how much

[13] A. Charnes, W. W. Cooper and B. Mellon, "Blending Aviation Gasolines," *Econometrica* Apr. 1952, XX, 135-59.

[14] A. Charnes, W. W. Cooper, and Donald Farr and Staff, "Linear Programming and Profit Preference Scheduling for a Manufacturing Firm," *Jour. Operations Research Society of America,* May 1953, I, 114-29.

newsprint to ship from each mill to each customer so as, first, to meet the contract requirements of each customer, second, to stay within the capacity limits of each mill, and third, to keep the aggregate freight bill as small as possible. This problem involves 1,200 variables (6 mills $\times$ 200 customers), in contrast to the two or four variable problems we have been discussing. In the final solution most of these variables will turn out to be zero—the question is which ones. This problem is solved by mathematical programming and, though formidable, is not really as formidable as the count of variables might indicate.

These few illustrations should suffice to indicate that mathematical programming is a practical tool for business planning. They show, also, that it is a flexible tool because both examples deviated from the format of the example used in our exposition. The petroleum application had the added feature of quality specification. In the newsprint application there were limits on the quantity of output as well as on the quantities of the inputs. Nevertheless mathematical programming handled them both easily.

On the other hand, it should be noted that both of these were small-scale applications, dealing with a single phase of the operation of a single firm. I believe that this has been true of all successful applications to date. Mathematical programmers are still a long way from solving the broad planning problem of entire industries or an entire economy. But many such broad problems are only enlarged versions of problems which have been met and solved in the context of the single firm. It is no longer premature to say that mathematical programming has proved its worth as a practical tool for finding optimal economic programs.

VII. *Conclusion*

Our objective has been only to introduce the basic notions of mathematical programming and to invest them with plausibility and meaning. The reader who would learn to solve a programming problem—even the simplest—will have to look elsewhere,[15] though this paper may serve as a useful background.

Although methods of solution have been omitted from this exposition, we must emphasize that these methods are fundamental to the whole concept of mathematical programming. Some eighty years ago Walras conceived of production in very much the same manner as mathematical

[15] The standard reference is T. C. Koopmans, ed., *Activity Analysis of Production and Allocation* (New York, 1951). Less advanced treatments may be found in A. Charnes, W. W. Cooper, and A. Henderson, *An Introduction to Linear Programming* (New York, 1953); and my own *Application of Linear Programming to the Theory of the Firm* (Berkeley, 1951).

programmers, and more recently A. Wald and J. von Neumann used this view of production and methods closely allied to those of mathematical programming to analyze the conditions of general economic equilibrium.[16] These developments, however, must be regarded merely as precursors of mathematical programming. Programming had no independent existence as a mode of economic analysis until 1947 when G. B. Dantzig announced the "simplex method" of solution which made practical application feasible.[17] The existence of a method whereby economic optima could be explicitly calculated stimulated research into the economic interpretation of mathematical programming and led also to the development of alternative methods of solution. The fact that economic and business problems when formulated in terms of mathematical programming can be solved numerically is the basis of the importance of the method. The omission of methods of solution from this discussion should not, therefore, be taken to indicate that they are of secondary interest.

We have considered only a few of the concepts used in mathematical programming and have dealt with only a single type of programming problem. The few notions we have considered, however, are the basic ones; all the rest of mathematical programming is elaboration and extension of them. It seems advisable to mention two directions of elaboration, for they remove or weaken two of the most restrictive assumptions which have here been imposed.

The first of these extensions is the introduction of time into the analysis. The present treatment has dealt with a single production period in isolation. But in many cases, successive production periods are interrelated. This is so, for example, in the case of a vertically integrated firm where the operation of some processes in one period is limited by the levels of operation in the preceding period of the processes which supply their raw materials. Efficient methods for analyzing such "dynamic" problems are being investigated, particularly by George Dantzig.[18] Although the present discussion has been static, the method of analysis can be applied to problems with a time dimension.

[16] Walras' formulation is in *Éléments d'économie politique pure ou théorie de la richesse sociale,* 2d ed. (Lausanne, 1889), 20^e Leçon. The contributions of A. Wald and J. von Neumann appeared originally in *Ergebnisse eines mathematischen Kolloquiums,* Nos. 6, 7, 8. Wald's least technical paper appeared in *Zeitschrift für Nationalökonomie,* VII (1936) and has been translated as "On some Systems of Equations of Mathematical Economics," *Econometrica,* Oct. 1951, XIX, 368-403. Von Neumann's basic paper appeared in translation as "A Model of General Economic Equilibrium," *Rev. Econ. Stud.,* 1945-46, XIII, 1-9.

[17] G. B. Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities," T. C. Koopmans, ed., *op. cit.,* pp. 339-47.

[18] "A Note on a Dynamic Leontief Model with Substitution" (abstract), *Econometrica,* Jan. 1953, XXI, 179.

The second of these extensions is the allowance for changes in the prices of factors and final products. In our discussion we regarded all prices as unalterable and independent of the actions of the economic unit under consideration. Constant prices are, undeniably, a great convenience to the analyst, but the method can transcend this assumption when necessary. The general mathematical theory of dealing with variable prices has been investigated[19] and practical methods of solution have been developed for problems where the demand and supply curves are linear.[20] The assumption of constant prices, perhaps the most restrictive assumption we have made, is adopted for convenience rather than from necessity.

Mathematical programming has been developed as a tool for economic and business planning and not primarily for the descriptive, and therefore predictive, purposes which gave rise to the marginal analysis. Nevertheless it does have predictive implications. In so far as firms operate under the conditions assumed in mathematical programming it would be unreasonable to assume that they acted as if they operated under the conditions assumed by the marginal analysis. Consider, for example, the automobile firm portrayed in Figure 1. How would it respond if the price of automobiles were to fall, say by $50 a unit? In that case the net revenue per automobile would be $250, the same as the net revenue per truck. Diagrammatically, the result would be to rotate the lines of equal revenue until their slope was 45 degrees. After this rotation, point C would still be optimum and this change in prices would cause no change in optimum output. Mathematical programming gives rise, thus, to a kinked supply curve.

On the other hand, suppose that the price of automobiles were to rise by $50. Diagrammatically this price change would decrease the steepness of the equal revenue lines until they were just parallel to the metal stamping line. The firm would then be in a position like that illustrated by the YY′ line in Figure 5. The production plans corresponding to points on the line segment DC in Figure 1 would all yield the same net revenue and all would be optimal. If the prices of automobiles were to rise by more than $50 or if a $50 increase in the price of automobiles were accompanied by any decrease in the price of trucks, the point of optimal production would jump abruptly from point C to point D.

Thus mathematical programming indicates that firms whose choices

[19] See H. W. Kuhn and A. W. Tucker, "Non-Linear Programming," in J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley, 1951), pp. 481-92.

[20] I reported one solution of this problem to a seminar at the Massachusetts Institute of Technology in September 1952. Other solutions may be known.

are limited to distinct processes will respond discontinuously to price variations: they will be insensitive to price changes over a certain range and will change their levels of output sharply as soon as that range is passed. This theoretical deduction surely has real counterparts.

The relationship between mathematical programming and welfare economics is especially close. Welfare economics studies the optimal organization of economic effort; so does mathematical programming. This relationship has been investigated especially by Koopmans and Samuelson.[21] The finding, generally stated, is that the equilibrium position of a perfectly competitive economy is the same as the optimal solution to the mathematical programming problem embodying the same data.

Mathematical programming is closely allied mathematically to the methods of input-output analysis or interindustry analysis developed largely by W. W. Leontief.[22] The two methods were developed independently, however, and it is important to distinguish them conceptually. Input-output analysis finds its application almost exclusively in the study of general economic equilibrium. It conceives of an economy as divided into a number of industrial sectors each of which is analogous to a process as the term is used in mathematical programming. It then takes either of two forms. In "open models" an input-output analysis starts with some specified final demand for the products of each of the sectors and calculates the level at which each of the sector-processes must operate in order to meet this schedule of final demands. In "closed models" final demand does not appear but attention is concentrated on the fact that the inputs required by each sector-process must be supplied as outputs by some other sector-processes. Input-output analysis then calculates a mutually compatible set of output levels for the various sectors. By contrast with mathematical programming the conditions imposed in input-output analysis are sufficient to determine the levels of the processes and there is no scope for finding an optimal solution or a set of "best" levels. To be sure, input-output analysis can be regarded as a special case of mathematical programming in which the number of products is equal to the number of processes. On the other hand, the limitations on the supplies of resources which play so important a rôle in mathematical programming are not dealt with explicitly in input-output analysis. On the whole it seems

[21] T. C. Koopmans, "Analysis of Production as an Efficient Combination of Activities," in T. C. Koopmans, ed., *op. cit.*, pp. 33-97; P. A. Samuelson, "Market Mechanisms and Maximization" (a paper prepared for the Rand Corp., 1949).

[22] W. W. Leontief, *The Structure of American Economy 1919-1939,* 2nd. ed. (New York, 1951).

best to regard these two techniques as allied but distinct methods of analysis addressed to different problems.

Mathematical programming, then, is of significance for economic thinking and theory as well as for business and economic planning. We have been able only to allude to this significance. Indeed, apart from the exploration of welfare implications, very little thought has been given to the consequences for economics of mathematical programming because most effort has been devoted to solving the numerous practical problems to which it gives rise. The outlook is for fruitful researches into both the implications and applications of mathematical programming.